

Notes on Orthogonal Frequency Division Multiplexing (OFDM)

1. Discrete Fourier Transform

As a reminder, the analytic forms of Fourier and inverse Fourier transforms are

$$X(f) = \int_{-\infty}^{\infty} x(t) \psi(t, f) dt = \int_{-\infty}^{\infty} x(t) \exp(-j2\pi ft) dt \quad (1.1)$$

where we have assumed that Fourier transform direction is from time axis (domain) to frequency axis (domain). In this sense $x(t)$ is our time signal, while $X(f)$ is the frequency signal or rather the frequency spectrum of $x(t)$. Finally $\psi(t, f) = \exp(-j2\pi ft)$ is the operator that enables the act of transformation from time axis to frequency axis. The inverse Fourier transform will be given by

$$x(t) = \int_{-\infty}^{\infty} X(f) \psi(f, t) df = \int_{-\infty}^{\infty} X(f) \exp(j2\pi ft) df \quad (1.2)$$

(1.2) allows the return to the time axis from the frequency axis, where the operator is now $\psi(f, t) = \psi^*(t, f) = \exp(j2\pi ft)$, i.e. the conjugate of the operator in (1.1). Symbolically (1.1) and (1.2) can be represented as below

$$X(f) = F[x(t)] \quad , \quad x(t) = F^{-1}[X(f)] \quad (1.3)$$

Some comments are in order regarding Fourier transform and its inverse

- In general Fourier transform is applicable to transformations between any axes and domains, for instance to go from spatial axis to spatial frequency axis, again Fourier transform and its inverse can be used. In these lecture notes, our axes will continue to be time and frequency as given in (1.1) and (1.2).
- Fourier transform can be multidimensional. It means that the single integrals in (1.1) and (1.2) can be extended to double or triple or N number of integrals.
- As shown by (1.1) and (1.2), to include all behaviours along one axis, the integral limits range from $-\infty$ to $+\infty$. In general these limits will be finite.
- The transformation operators have the property that $|\psi(f, t)| = |\psi(t, f)| = 1$. Thus no scaling is applied during transformations. This way, we can expect that the energies or the powers (of the same signal) in the two domains to be exactly equivalent.

We now introduce the discrete Fourier transform by associating the time and frequency functions, variables and the operators as follows

$$\begin{aligned} t \rightarrow n \quad , \quad f \rightarrow k \quad , \quad x(t) \rightarrow x(n) \quad , \quad X(f) \rightarrow X(k) \\ \psi(t, f) \rightarrow \psi(n, k) \quad , \quad \psi(f, t) \rightarrow \psi(k, n) \end{aligned} \quad (1.4)$$

n and k will be integer variables, limited to the range

$$0 \leq n \leq N-1 \quad , \quad 0 \leq k \leq N-1 \quad (1.5)$$

Therefore N may be regarded to represent the infinity of the discrete world. Bearing in mind that integration of the analytic world will correspond to summation in discrete domain, the discrete Fourier transform (DFT) of $x(n)$ into $X(k)$ will be

$$X(k) = \sum_{n=0}^{N-1} x(n) \exp(-j2\pi kn/N) \quad (1.6)$$

$1/N$ in the argument of the exponential operator in (1.6) acts as normalization factor. The inverse discrete Fourier transform (IDFT) of $X(k)$ into $x(n)$ will be

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \exp(j2\pi kn/N) \quad (1.7)$$

Then symbolically, (1.6) and (1.7) can be combined as

$$x(n) \xrightleftharpoons[\text{IDFT}]{\text{DFT}} X(k) \quad (1.8)$$

In analytic world, (1.1) and (1.2) work without any problems. When switching to discrete case, care must be taken, because there are three main points to observe there

- a) Infinity of analytic world must suitably be converted to the appropriate lengths in discrete world.
- b) Grid spacing of the two domains of the discrete world must be taken into account. In analytic case this is properly covered since dt and df correspond to the same infinitesimally small intervals, thus creating no scaling problems.
- c) The rules of sampling theorem should be properly adhered to. This is because n and k are effectively the sampling rates in time and frequency domains. Hence the smallest time resolution that can be achieved is $2n$ and correspondingly the smallest frequency resolution that we can attain is $2k$.

Example 1.1 : Let $x(n)$ be a delta (impulse) function such that

$$x(n) = \begin{cases} 1 & n = 0 \\ 0 & 1 \leq n \leq N-1 \end{cases} \quad (1.9)$$

Find $X(k)$ for this discrete time delta function.

Solution : Graphically, (1.9) will be as shown in Fig. 1.1. Using (1.6), we get

$$\begin{aligned}
 X(k) &= \sum_{n=0}^{N-1} x(n) \exp(-j2\pi kn / N) \\
 &= \overbrace{x(n=0)}^1 \times \exp(0) + \overbrace{x(n=1)}^0 \times \exp(-j2\pi k / N) + \overbrace{x(n=2)}^0 \times \exp(-j4\pi k / N) \\
 &\quad + \cdots + \overbrace{x(n=N-1)}^0 \times \exp(-j\pi k) = 1
 \end{aligned}
 \tag{1.10}$$

So the result is $X(k) = 1$ which agrees with the analytic result of

$$X(f) = F[x(t) = \delta(t)] = 1 \tag{1.11}$$

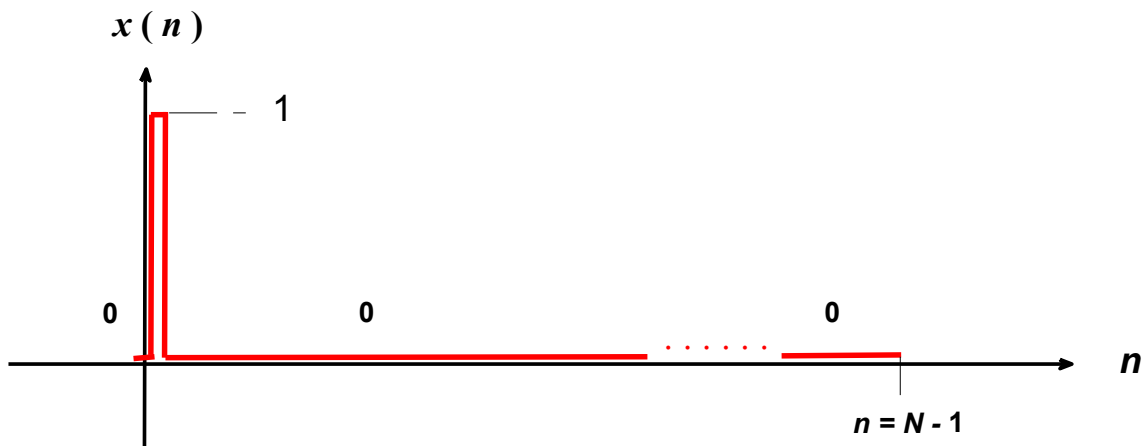


Fig. 1.1 The graph of discrete (time) delta function given by (1.9).

Fig. 1.2 displays the graph of $X(k) = 1$.

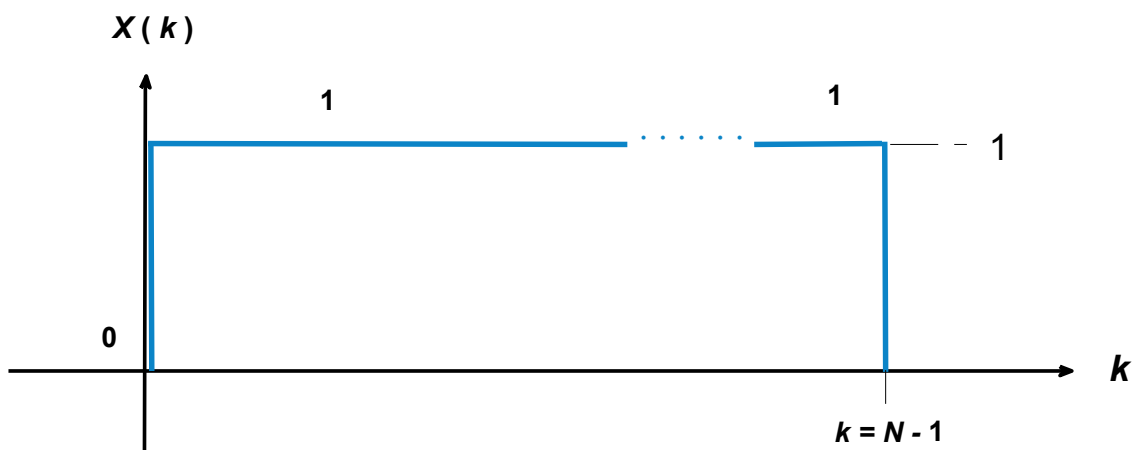


Fig 1.2 Discrete Fourier transform of discrete time delta function of (1.9).

Example 1.2 : Let $x(n)$ be time shifted a delta function such that

$$x(n) = \begin{cases} 1 & n = m \\ 0 & \text{elsewhere} \end{cases} \quad (1.12)$$

Find $X(k)$ for this discrete time delta function.

Solution 1.2 : Graphically (1.12) will be as shown in Fig. 1.3. Again using the general formulation given in (1.6), we get

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x(n) \exp(-j2\pi kn / N) \\ &= \overbrace{x(0)}^0 \times \exp(0) + \overbrace{x(1)}^0 \times \exp(-j2\pi k / N) + \overbrace{x(2)}^0 \times \exp(-j4\pi k / N) \\ &+ \dots + \overbrace{x(m-1)}^0 \exp[-j2\pi k(m-1) / N] + \overbrace{x(m)}^1 \exp(-j2\pi km / N) + \overbrace{x(m+1)}^0 \exp[-j2\pi k(m+1) / N] \\ &+ \dots + \overbrace{x(N-1)}^0 \times \exp(-j\pi k) = \exp(-j2\pi km / N) \end{aligned} \quad (1.13)$$

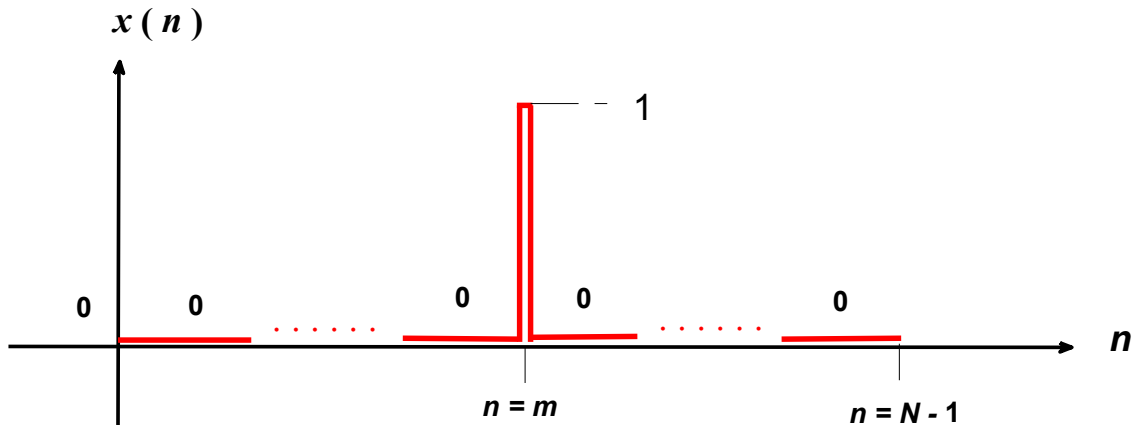


Fig. 1.3 The graph of discrete time shifted delta function given by (1.12).

Again the finding of (1.13) agrees with the analytic result of

$$X(f) = F[x(t) = \delta(t - t_1)] = \exp(-j2\pi ft_1) \quad (1.14)$$

Since (1.13) is complex, it can be plotted in two parts as split below

$$X(k) = \exp(-j2\pi km / N) \rightarrow \begin{cases} |X(k)| = 1 \\ \Theta(k) = -2\pi km / N \end{cases} \quad (1.15)$$

The plot associated with (1.15) is given in Fig. 1.4.

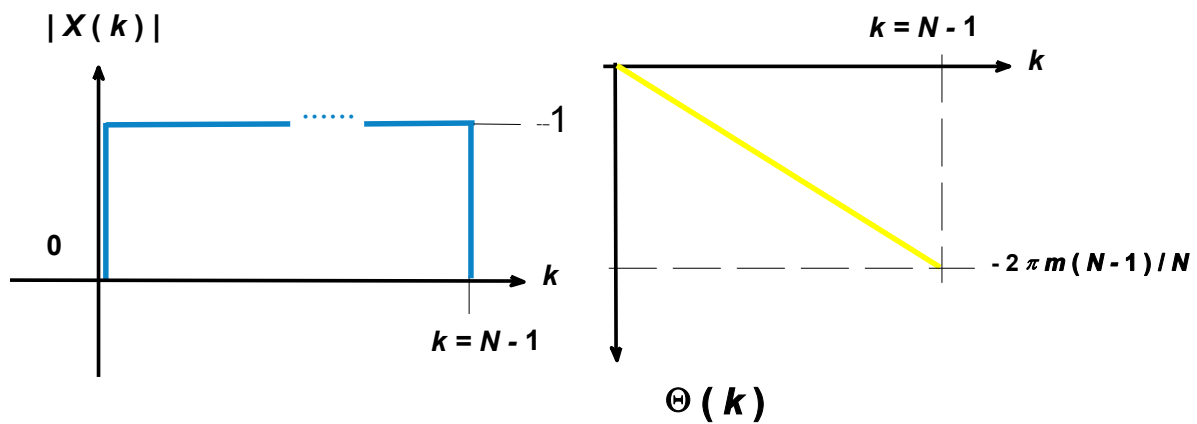


Fig. 1.4 Discrete Fourier transform of time shifted delta function (from (1.15)).

Example 1.3 : Now take a single exponential at a specific frequency $k = m$, thus we may write

$$x(n) = \exp(j2\pi mn / N) \tag{1.16}$$

So we try to find the discrete Fourier transform of $x(n)$ as given in (1.16).

Solution 1.3 : Again by using (1.6), we get

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x(n) \exp(-j2\pi kn / N) = \sum_{n=0}^{N-1} \exp(j2\pi mn / N) \exp(-j2\pi kn / N) \\ &= \sum_{n=0}^{N-1} \exp[j2\pi n(m-k) / N] \end{aligned} \tag{1.17}$$

By using the identity

$$\sum_{n=0}^{N-1} \exp[j2\pi n(m-k) / N] = \begin{cases} N & \text{if } m-k = pN, \quad p \rightarrow \text{integer} \\ 0 & \text{if } m \neq k, \quad m \rightarrow \text{integer}, \quad k \rightarrow \text{integer} \end{cases} \tag{1.18}$$

$X(k)$ given by (1.17) becomes

$$X(k) = \begin{cases} N & \text{if } k = m \\ 0 & \text{otherwise} \end{cases} \tag{1.19}$$

(1.19) means a delta function at $k = m$. The corresponding relation of the analytic world is

$$X(f) = F[x(t) = \exp(-j2\pi f_1 t)] = \delta(f - f_1) \tag{1.20}$$

Fig. 1.5 displays the graph of (1.19).

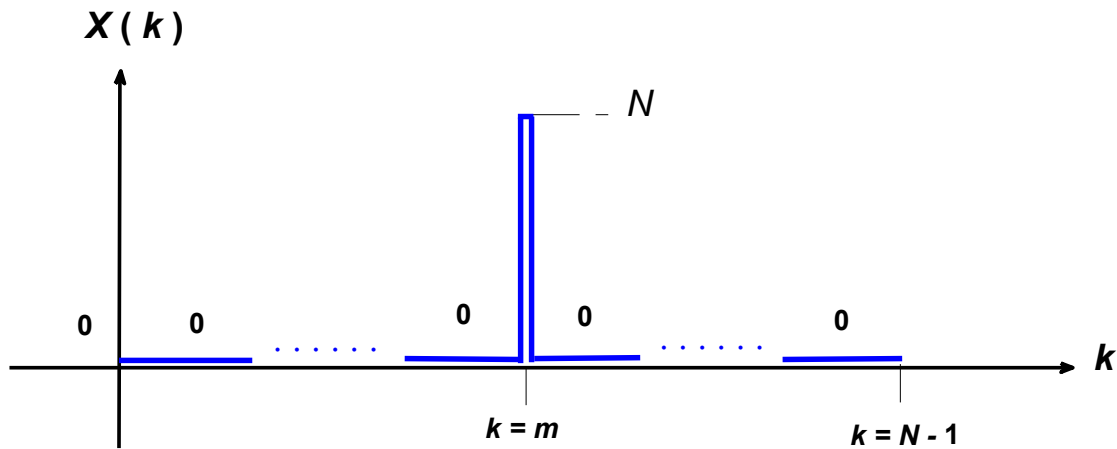


Fig. 1.5 Discrete Fourier transform of a single exponential as given by (1.16).

Exercise 1.1 : Prove the identity in (1.18). Note that if m and k are limited to a single N range, so that $0 \leq m \leq N-1$ and $0 \leq k \leq N-1$, then (1.18) can be converted into

$$\sum_{n=0}^{N-1} \exp[j2\pi n(m-k)/N] = \begin{cases} N & \text{if } m = k, \quad 0 \leq m \leq N-1 \\ 0 & \text{if } m \neq k, \quad 0 \leq k \leq N-1 \end{cases} \quad (1.21)$$

Exercise 1.2 : Find the discrete Fourier Transform of the sum of two exponentials as given below.

$$x(n) = A_1 \exp(j2\pi m_1 n / N) + A_2 \exp(j2\pi m_2 n / N) \quad (1.22)$$

Check that the discrete Fourier Transform that you have found for (1.22) resembles the corresponding analytic expression.

2. Basis for Orthogonal Frequency Division Multiplexing (OFDM)

Conventionally we place a message signal on a single carrier. For instance, to have double side band AM type of modulation, if $s(t)$ is the message signal, we simply multiply by a sinusoidal at the frequency of f_c to obtain the modulated signal as

$$y(t) = s(t) \exp(j2\pi f_c t) \quad \text{or} \quad y(t) = s(t) \cos(2\pi f_c t) \quad \text{or} \quad y(t) = s(t) \sin(2\pi f_c t) \quad (2.1)$$

Due to the frequency spectrum of $s(t)$, i.e., $S(f)$ occupying a finite bandwidth, we have to take into account, the frequency response of the channel for the transmitted signal to arrive at the receiver without distortions.

Suppose that $S(f)$ has a bandwidth of $2W$, then $Y(f) = F[y(t)]$ has the two sided spectrum of $2W$ each centered around $f = |f_c|$ as shown in Fig. 2.1. Assuming that the communication channel that our modulated signal is going to pass through has the following frequency response

$$C(f) = \begin{cases} 1 \text{ or any constant } < 1 & \text{if } |f - f_c| \leq W \\ \text{arbitrary} & \text{elsewhere} \end{cases} \quad (2.1)$$

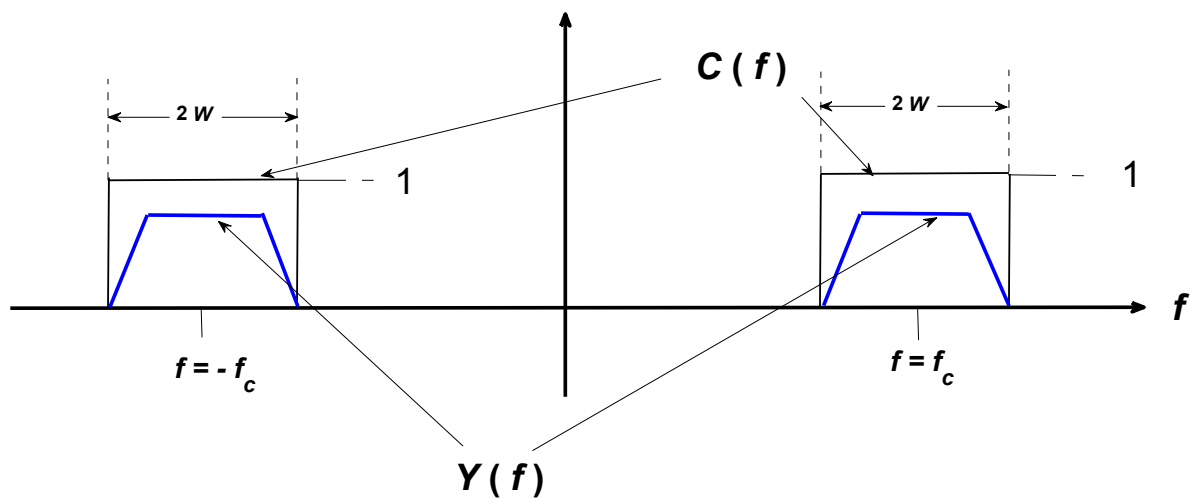


Fig. 2.1 A communication channel that introduces no distortion.

then confined to the frequency range of $|f - f_c| \leq W$ at the output of the channel, we will have

$$r(t) = F^{-1}[C(f)Y(f)] = F^{-1}[1 \times Y(f)] = y(t) \quad (2.2)$$

The implication of (2.2) is that we have been able to receive the exact copy of the transmitted signal and no distortions due to channel impairments have been experienced. Note (2.2) is as a result of the channel response being frequency independent within the band of our transmitted signal.

Next, let's assume that our transmitted signal remains the same, but the channel response turns into the one shown in Fig. 2.2.

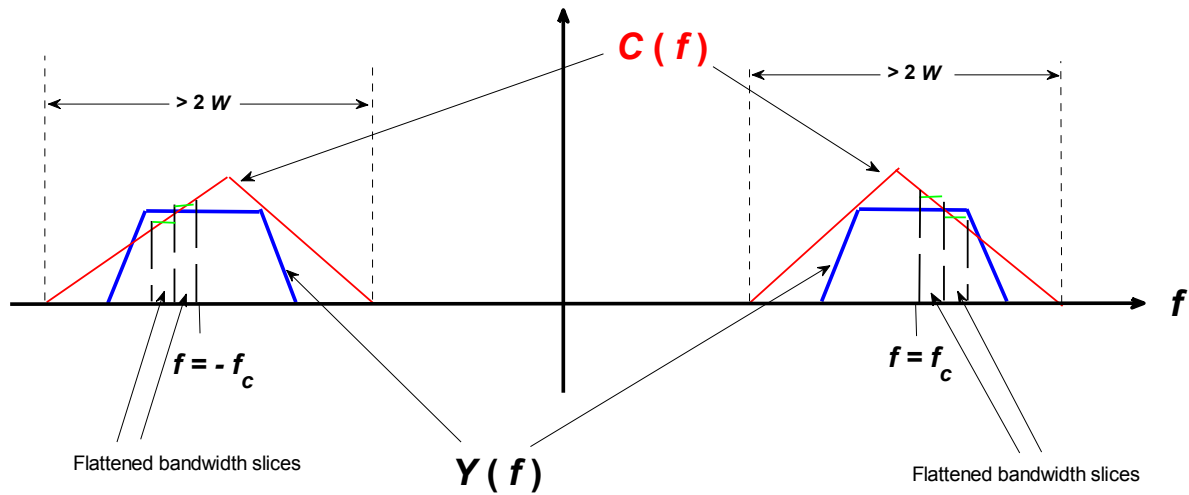


Fig. 2.2 A communication channel that will introduce distortion.

Note that in the case of Fig. 2.2, the spectrum of the transmitted signal is still accommodated within the channel response, but since $C(f)$ no longer satisfies (2.1), the transmitted signal will definitely be received definitely with some distortion. But if we imagine that we slice the channel into finer frequency intervals, then it is possible to obtain nearly flat channel (frequency independent) responses like the one given in (2.1). For each slice, we can assign a different carrier rather than using a one single carrier centered at f_c . Slicing into the source signal band into narrower frequency slots means that we can use extended symbol durations.

3. Construction of Orthogonal Frequency Division Multiplexing (OFDM)

OFDM is historically based on frequency shift keying (FSK). In OFDM, the number of carriers called subcarriers, are too many and they are arranged to be orthogonal over one symbol interval.

For the derivation of the basic relation for orthogonality, we first consider the following integral covering two sinusoidal signals

$$I = \int_0^T \cos(2\pi f_p t + \phi_p) \cos(2\pi f_m t + \phi_m) dt \quad (3.1)$$

It is easy by hand derivation that the integral in (3.1) will evaluate to zero under the following conditions

$$\begin{aligned} f_p &= \frac{p}{T}, \quad f_m = \frac{m}{T} & p, m &\rightarrow \text{integers} \\ \phi_p, \phi_m &\rightarrow \text{arbitrary} \end{aligned} \quad (3.2)$$

(3.2) can also be interpreted as

$$f_p - f_m = \frac{p-m}{T}$$

$$T_p = \frac{1}{f_p}, \quad T_m = \frac{1}{f_m}, \quad T_m - T_p = T \left(\frac{1}{m} - \frac{1}{p} \right) \quad (3.3)$$

The proof of (3.1) becoming zero with the setting of (3.2) is given below

using $\cos(A)\cos(B) = 0.5[\cos(A-B) + \cos(A+B)]$

$$I = 0.5 \int_0^T \cos[2\pi(f_p - f_m)t + \phi_p - \phi_m] dt + 0.5 \int_0^T \cos[2\pi(f_p + f_m)t + \phi_p + \phi_m] dt$$

$$= 0.5 \frac{\sin[2\pi(f_p - f_m)t + \phi_p - \phi_m]}{2\pi(f_p - f_m)} \Big|_0^T + 0.5 \frac{\sin[2\pi(f_p + f_m)t + \phi_p + \phi_m]}{2\pi(f_p + f_m)} \Big|_0^T$$

$$= 0.5 \frac{\sin[2\pi(p-m) + \phi_p - \phi_m] - \sin(\phi_p - \phi_m)}{2\pi(f_p - f_m)} + 0.5 \frac{\sin[2\pi(p-m) + \phi_p + \phi_m] - \sin(\phi_p + \phi_m)}{2\pi(f_p + f_m)}$$

$$= 0 + 0 \quad (3.4)$$

Note that the validity of (3.4) can also be shown in Matlab as follows

```
syms Fip Fim t
T = 0.1; p = 5; m = 2; fp = p/T; fm = m/T;
I = int(cos(2*pi*fp*t + Fip)*cos(2*pi*fm*t + Fim), t, 0, T)
```

After running this piece of Matlab code, we get the result of $I = 0$.

(3.4) can also be established using exponentials in a shorter way.

$$I = \int_0^T \exp(j2\pi f_p t) \exp(j\phi_p) \exp(-j2\pi f_m t) \exp(-j\phi_m) dt$$

$$= \exp[j(\phi_p - \phi_m)] \int_0^T \exp[j2\pi(f_p - f_m)t] dt = T \exp[j(\phi_p - \phi_m)] \quad \text{if } f_p = f_m \text{ or } p = m$$

$$= \exp[j(\phi_p - \phi_m)] \frac{\exp[j2\pi(f_p - f_m)t]}{j2\pi(f_p - f_m)} \Big|_0^T = \exp[j(\phi_p - \phi_m)] \frac{\exp[j2\pi(f_p - f_m)T] - 1}{j2\pi(f_p - f_m)}$$

$$= \exp[j(\phi_p - \phi_m)] \frac{\overbrace{\exp[j2\pi(p-m)]}^{=1 \text{ for } p-m \rightarrow \text{integer}} - 1}{j2\pi(f_p - f_m)} = 0 \quad \text{if } p \neq m \quad (3.5)$$

Note that for the complex case, orthogonality integral is written by taking the complex conjugate of the second function.

The implications of (3.2) to (3.5) are that if $\cos(2\pi f_p t)$ and $\cos(2\pi f_m t)$ are to be selected as OFDM subcarriers, then the orthogonality condition stated in (3.2) or (3.3) must hold with the relative phase difference between the subcarriers being unimportant.

Example 3.1 : In Fig. 3.1, we illustrate the simple case of four OFDM subcarriers

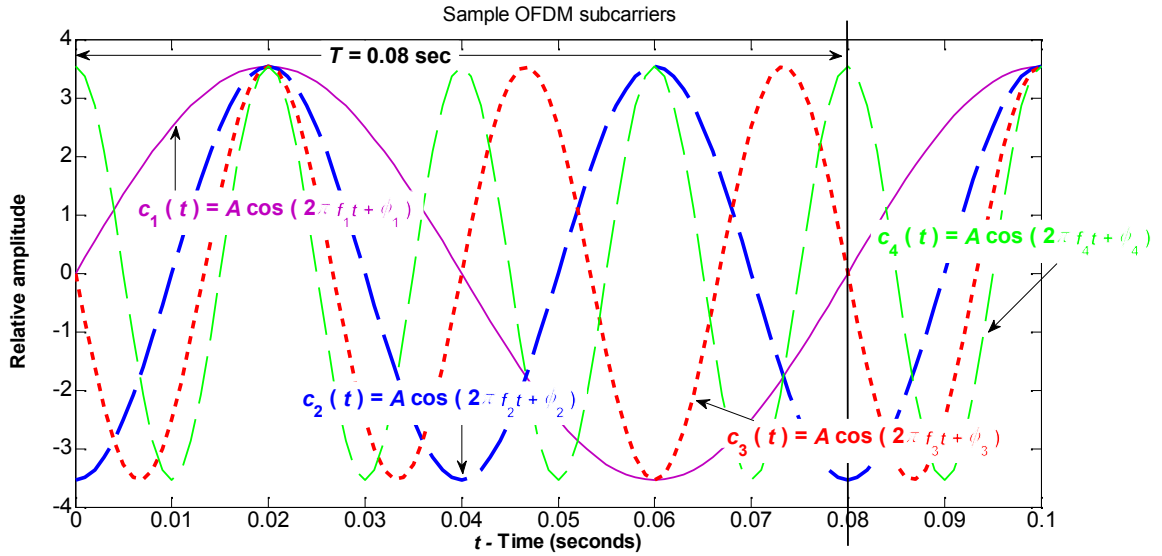


Fig. 3.1 Four sample OFDM subcarriers.

We can compile the following numeric data from Fig. 3.1

$$T = 0.08 \text{ sec} = 80 \text{ msec}$$

$$\text{First subcarrier : } c_1(t) = A \cos(2\pi f_1 t + \phi_1) , f_1 = 12.5 \text{ Hz} = 1/T , \phi_1 = -\pi/2$$

$$\text{Second subcarrier : } c_2(t) = A \cos(2\pi f_2 t + \phi_2) , f_2 = 25 \text{ Hz} = 2/T , \phi_2 = \pm\pi$$

$$\text{Third subcarrier : } c_3(t) = A \cos(2\pi f_3 t + \phi_3) , f_3 = 37.5 \text{ Hz} = 3/T , \phi_3 = \pi/2$$

$$\text{Fourth subcarrier : } c_4(t) = A \cos(2\pi f_4 t + \phi_4) , f_4 = 50 \text{ Hz} = 4/T , \phi_4 = 0 \quad (3.6)$$

As seen from (3.6) and Fig. 3.1, the OFDM subcarriers despite their differing (initial) phases, $c_1(t)$, $c_2(t)$, $c_3(t)$, $c_4(t)$ have complete (integer) number of cycles within one whole symbol duration, T . Similarly the differences between the frequencies of the subcarriers are integer multiples of the inverse of the symbol duration, T . This way, orthogonality conditions of (3.2) and (3.3) are perfectly satisfied by $c_1(t)$, $c_2(t)$, $c_3(t)$, $c_4(t)$.

Exercise 3.1 : Using the Matlab file `orthogonality_syms.m` or otherwise, confirm that the subcarriers $c_1(t)$, $c_2(t)$, $c_3(t)$, $c_4(t)$ do indeed satisfy the orthogonality, when any of the two subcarriers are integrated over the interval T , that is

$$\int_0^T c_p(t) c_m(t) dt = 0 , 1 \leq p \leq 4 , 1 \leq m \leq 4 , p \neq m \quad (3.7)$$

At the transmitter side OFDM signal will be obtained by modulating the OFDM subcarriers by a PSK or QAM signals. The simplified block of an OFDM transmitter is given in Fig. 3.2.

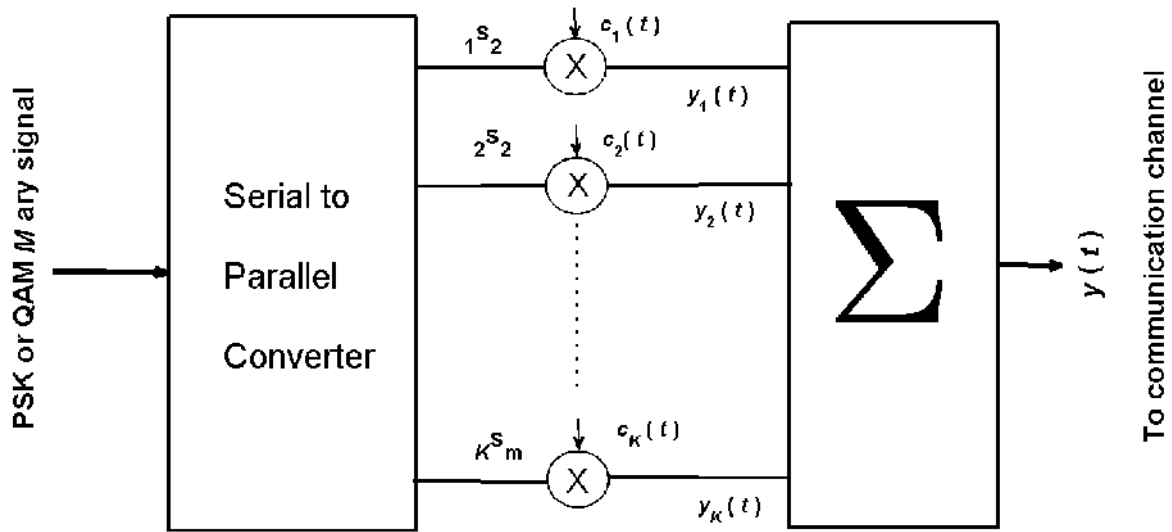


Fig. 3.2 Simplified block diagram of an OFDM transmitter.

As seen in Fig. 3.2 for an M ary PSK or QAM, K number of subcarriers is reserved. It could be that $K = M$, which will be the setting in these notes. To exemplify the operations carried out in Fig. 3.2, we take the constellation diagram of 4 PSK and a typical flow of the first four PSK symbols along time axis as illustrated in Fig. 3.3.

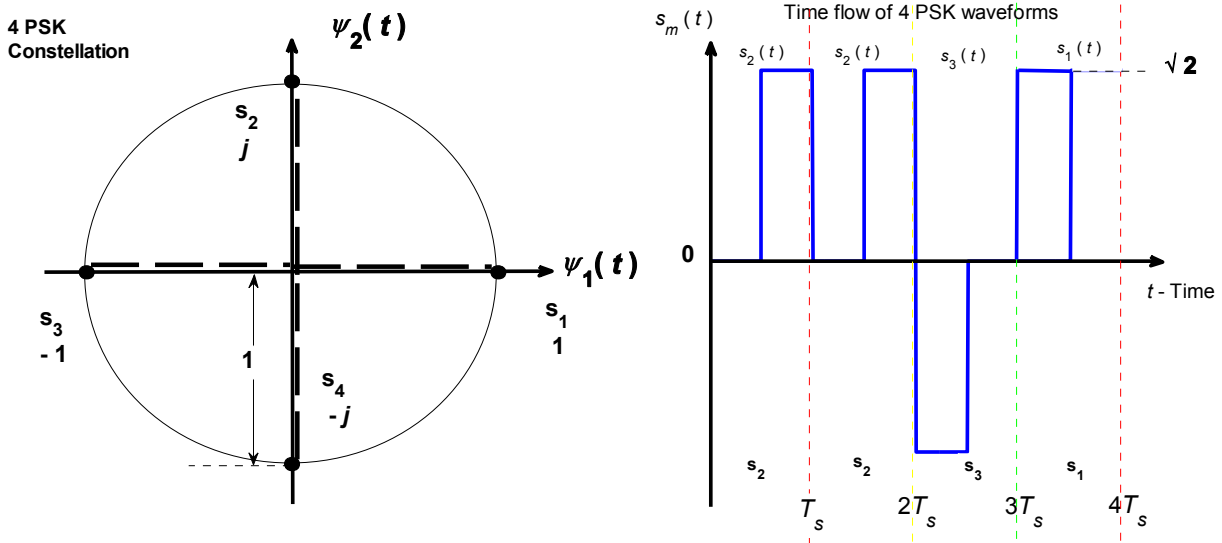


Fig. 3.3 4 PSK constellation and related first four symbols.

Note that on the constellation diagram of Fig. 3.3, we also show the Matlab notation of numeric values for the 4 PSK signals, $s_1 \dots s_4$. These are

$$s_1 = 1, s_2 = j, s_3 = -1, s_4 = -j \quad (3.8)$$

As indicated in Fig. 3.2, modulation of subcarriers will be established by the multiplication of the PSK symbols of (3.8) by the subcarriers. This way, k th modulated subcarrier will be obtained from

$$y_k(t) = {}_k \mathbf{s}_m c_k(t) = {}_k \mathbf{s}_m \cos(2\pi f_k t) \quad , \quad 1 \leq k \leq K \quad (3.9)$$

where K is the total number of subcarriers. We show in Fig. 3.4 how the four PSK signals of Fig. 3.3 are assigned to the four OFDM subcarriers of $c_1(t) \cdots c_4(t)$, to constitute $y_1(t) \cdots y_4(t)$ with the initial phases of the subcarriers given in (3.6) all being zero. As demonstrated by Fig. 3.4, the assignment operation also allows the extension of symbol duration from T_s to $T = 4T_s$. This is equivalent to dividing the signal bandwidth into finer slices so that the response of the communication channel will appear nearly flat, i.e. the act illustrated in Fig. 2.1.

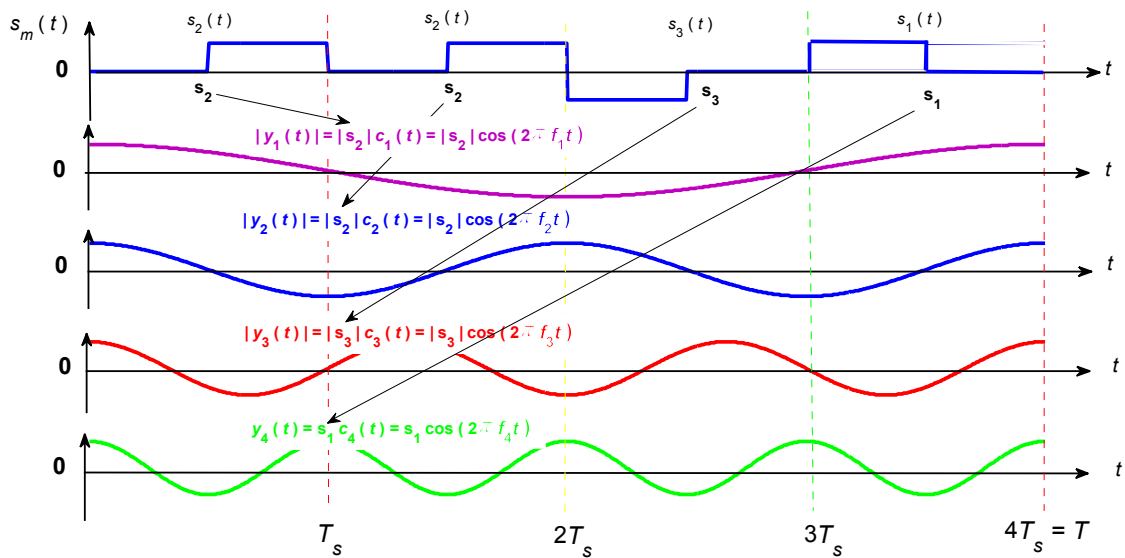


Fig. 3.4 The assignment of the four PSK symbols to the four OFDM subcarriers.

Bearing in mind that the OFDM subcarriers modulated by the PSK symbols are going to be added to each other within one OFDM symbol duration, i.e., within $T = MT_s$ as shown in Fig. 3.2, then we will have the following mathematical expression for the OFDM signal, $y(t)$

$$y(t) = \sum_{k=1}^K y_k(t) = \sum_{k=1}^K {}_k \mathbf{s}_m c_k(t) = \sum_{k=1}^K {}_k \mathbf{s}_m \cos(2\pi f_k t) \quad , \quad 1 \leq m \leq M \quad , \quad 0 \leq t \leq T \quad (3.10)$$

where ${}_k \mathbf{s}_m$ is one of the m th symbol of PSK, i.e., \mathbf{s}_m , placed onto the k subcarrier. Considering the numeric values of \mathbf{s}_m in (3.8), we see that $y(t)$ will be complex. Then $y(t)$ can be represented as

$$y(t) = y_r(t) + y_i(t) \quad y_r(t) \rightarrow \text{real part of } y(t) \quad , \quad y_i(t) \rightarrow \text{imaginary part of } y(t)$$

$$|y(t)| = [y_r^2(t) + y_i^2(t)]^{0.5} \quad \theta_y(t) = \tan^{-1} \left[\frac{y_i(t)}{y_r(t)} \right] \quad (3.11)$$

Thus, $|y(t)|$, $\theta_y(t)$ and their combination can be plotted as shown in Fig. 3.5, where some numeric values of the magnitude and the phase are given as well. Thus we can visualize $y(t)$ as a vector whose length (magnitude) and phase are changing with the progress of time.

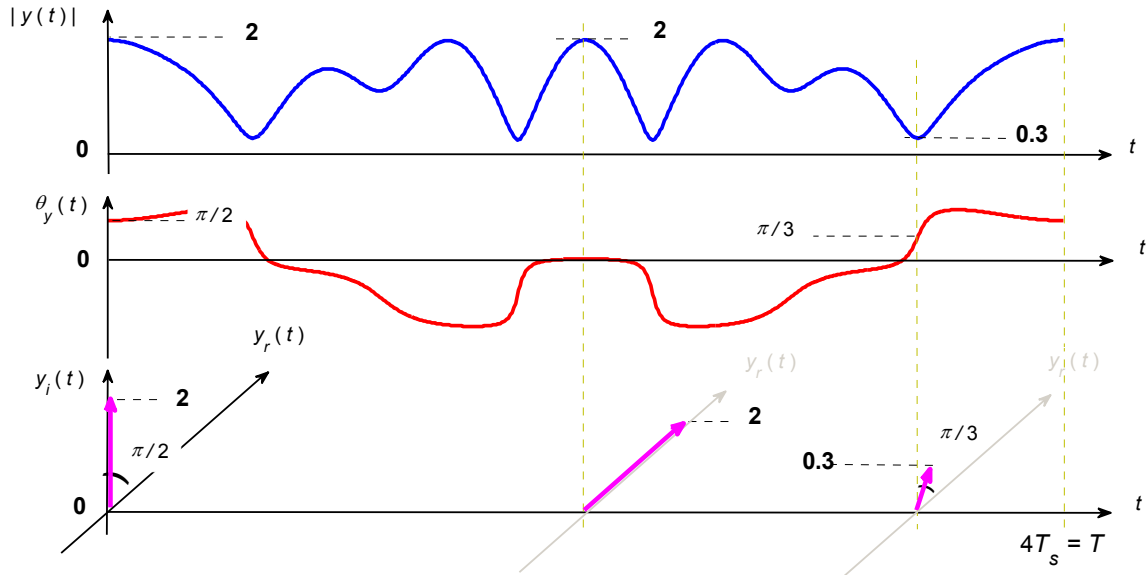


Fig. 3.5 The magnitude and phase plots of OFDM signal.

By converting the OFDM subcarriers into exponentials, (3.10) will become

$$y(t) = \sum_{k=1}^K s_k \exp(j2\pi f_k t) \quad , \quad 1 \leq k \leq K \quad , \quad 0 \leq t \leq T \quad (3.12)$$

Written in discrete Fourier transform notation, (3.12) will be

$$y(n) = \frac{1}{N} \sum_{k=1}^K s_k \exp(j2\pi f_k n / N) \quad (3.13)$$

where we have let $t \rightarrow n / N$. Comparing (3.13) to (1.7), we conclude that (3.12) is in the form of inverse discrete Fourier transform (IDFT). This means that the transmitter of Fig. 3.2 can implement the OFDM modulation via IDFT in a fast manner even if the number of subcarriers reaches numbers like 512. We know that the job of the receiver will be to demodulate and recover the symbols s_k placed on the subcarriers. From the relations given in (1.6) and (1.7), we expect that the demodulation of the message symbols can be performed by the discrete Fourier transform (DFT) operation, that is

$$d_k = \sum_{n=0}^{N-1} y(n) \exp(-j2\pi f_k n / N) \rightarrow s_k \quad (3.14)$$

We shall prove later that the operation in (3.14) will indeed deliver the s_k symbols.

It is instructive to examine the frequency spectrums of OFDM signal. For this, we select the simple case of 4 PSK and four OFDM subcarriers given in (3.6). As apparent from Fig. 3.4 and (3.8), the sinusoidal OFDM subcarriers are going to be modulated by rectangular shaped time limited (PSK) symbols. Therefore it is natural to anticipate that, the frequency spectrums of thus modulated OFDM subcarriers will be in the form of sinc functions as shown in Fig. 3.6. Here the relative phases introduced by PSK symbols are ignored. As seen from Fig. 3.6, the orthogonality condition embedded into these subcarriers ensures that at the spectral peak of one modulated subcarrier, the spectrums of all other subcarriers pass through zero. This point is important from the point of view of successful demodulation and detection of the transmitted PSK symbols.

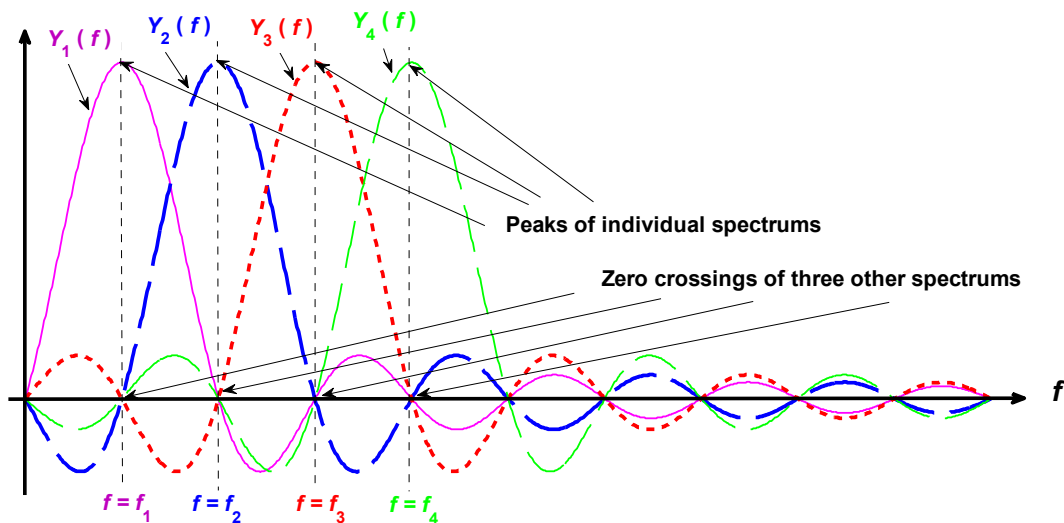


Fig. 3.6 Spectrums of modulated OFDM subcarriers.

We now move onto OFDM receiver. Fig. 3.7 gives the simplified block diagram of such a receiver.

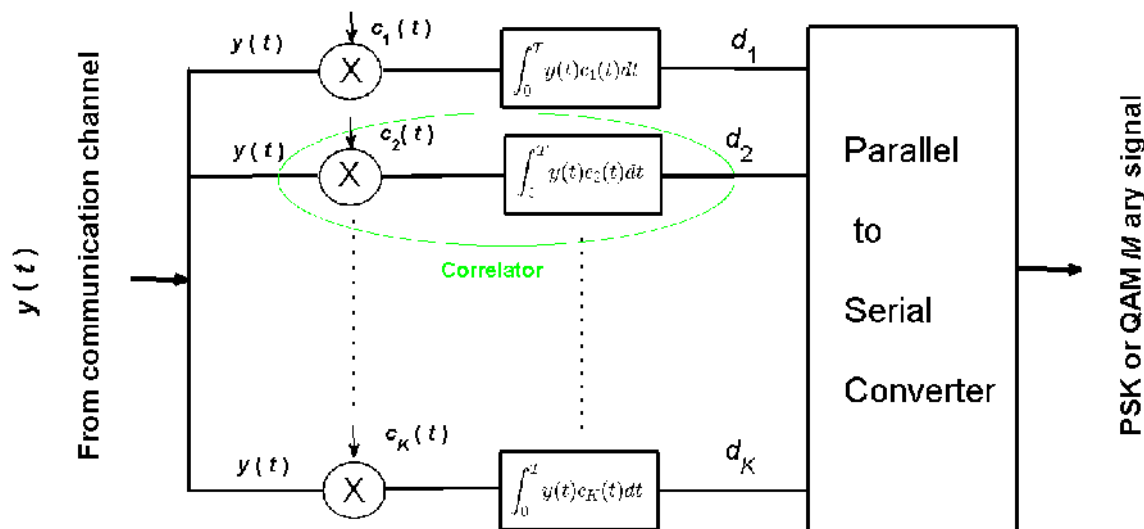


Fig. 3.7 Simplified block diagram of an OFDM receiver.

As understood from Fig. 3.7, the communication channel that exists between the transmitter of Fig. 3.2 and receiver of Fig. 3.7 is assumed to be band unlimited. The way, at the input of the,

receiver, there is an exact copy of the transmitted OFDM signal. On each arm of the receiver of Fig. 3.7, multiplication of the received signal by one of the subcarriers corresponds to the act of correlation. Thus the output of k th arm of the correlator is

$$d_k = \int_0^T y(t) \cos(2\pi f_k t) dt \quad (3.15)$$

It is easy to prove that the mathematical operation of (3.15) will deliver the individual PSK symbols placed on the subcarriers sequentially at the transmitter. Below we do this for the sample case of 4 PSK of Fig. 3.4.

Initially we write the complete expression of the transmitted signal with the PSK symbols assigned to the subcarriers as shown in Fig. 3.4.

$$\begin{aligned} y(t) &= \sum_{k=1}^4 y_k(t) = \sum_{k=1}^4 \mathbf{s}_k c_k(t) = \sum_{n=1}^4 \mathbf{s}_n \cos(2\pi f_n t) \\ &= \mathbf{s}_2 \cos(2\pi f_1 t) + \mathbf{s}_2 \cos(2\pi f_2 t) + \mathbf{s}_3 \cos(2\pi f_3 t) + \mathbf{s}_1 \cos(2\pi f_4 t) \\ &= j \cos(2\pi f_1 t) + j \cos(2\pi f_2 t) - \cos(2\pi f_3 t) + \cos(2\pi f_4 t) \end{aligned} \quad (3.16)$$

Now based on (3.15), we evaluate the correlator output for the first arm of the receiver in Fig. 3.7.

$$\begin{aligned} d_1 &= \int_0^T y(t) \cos(2\pi f_1 t) dt \\ &= \overbrace{j \int_0^T \cos(2\pi f_1 t) \cos(2\pi f_1 t) dt}^{I_1} + \overbrace{j \int_0^T \cos(2\pi f_2 t) \cos(2\pi f_1 t) dt}^{I_2} \\ &\quad - \overbrace{\int_0^T \cos(2\pi f_3 t) \cos(2\pi f_1 t) dt}^{I_3} + \overbrace{\int_0^T \cos(2\pi f_4 t) \cos(2\pi f_1 t) dt}^{I_4} \end{aligned} \quad (3.16)$$

Due to the selection of frequencies f_1, f_2, f_3, f_4 , the orthogonality of (3.7) is satisfied, thus

$$\begin{aligned} \int_0^T \cos(2\pi f_p t) \cos(2\pi f_m t) dt &= 0 \quad , \quad \text{if } p \neq m \quad \text{and} \quad f_p - f_m = \frac{p-m}{T} \\ \int_0^T \cos(2\pi f_p t) \cos(2\pi f_m t) dt &= 0.5T \quad , \quad \text{if } p = m \end{aligned} \quad (3.17)$$

then the individual integrals in (3.16) will be

$$\begin{aligned}
I_1 &= j \int_0^T \cos(2\pi f_1 t) \cos(2\pi f_1 t) dt = 0.5 j t \Big|_0^T + \overbrace{\frac{\sin(4\pi f_1 t)}{8\pi f}}^{=0, \text{ due to orthogonality}} \Big|_0^T = 0.5 j T \\
I_2 &= j \int_0^T \cos(2\pi f_2 t) \cos(2\pi f_1 t) dt = 0 \quad , \quad I_3 = - \int_0^T \cos(2\pi f_3 t) \cos(2\pi f_1 t) dt = 0 \\
I_4 &= \int_0^T \cos(2\pi f_4 t) \cos(2\pi f_1 t) dt = 0
\end{aligned} \tag{3.18}$$

So, eventually the output of the first arm of the correlator will become $d_1 = 0.5 j T = 0.5 s_2 T$ which means that we have been able to demodulate successfully the PSK symbol (the scaled version of) s_2 placed on the subcarrier $c_1(t)$.

It is easy to demonstrate that, the output of the remaining arms, i.e., d_2, d_3, d_4 , in the 4 PSK equivalent of Fig. 3.7 will deliver the other symbols placed onto subcarriers $c_2(t), c_3(t), c_4(t)$.

Exercise 3.2 : Bearing in mind the assignment of 4 PSK symbols in Fig. 3.4, prove that the correlation operation of Fig. 3.7 and (3.15) will correctly demodulate the symbols on the subcarriers of $c_2(t), c_3(t), c_4(t)$.

4. Orthogonal Frequency Division Multiplexing (OFDM) Analysis in Discrete Domain

Now we describe modulation and demodulation processes of OFDM in terms of discrete Fourier and inverse Fourier transforms. For this, we again take the simplified case of 4 PSK and write for the discrete equivalent of subcarriers $c_1(t), c_2(t), c_3(t), c_4(t)$

$$\begin{aligned}
c_1(n) &= \exp(j2\pi f_1 n / 4) \quad , \quad c_2(n) = \exp(j2\pi f_2 n / 4) \\
c_3(n) &= \exp(j2\pi f_3 n / 4) \quad , \quad c_4(n) = \exp(j2\pi f_4 n / 4)
\end{aligned} \tag{4.1}$$

In (4.1) we have intentionally changed the time range into $N = 4$, since the subcarriers are limited to that time range, which can also be denoted by $T = 4$. Keep in mind that in discrete applications, the frequencies can only take integer values, hence the appropriate choice is

$$f_1 = 1, f_2 = 2, f_3 = 3, f_4 = 4 \tag{4.2}$$

As seen from (4.1), the subcarriers are complex, hence, we revise the orthogonality test of (3.7) as follows

$$\int_0^T c_p(t) c_m^*(t) dt = 0 \quad , \quad 1 \leq p \leq 4 \quad , \quad 1 \leq m \leq 4 \quad , \quad p \neq m \tag{4.3}$$

where * takes into account the possibility of subcarriers being complex. The discrete equivalence of (4.3) is

$$\sum_{n=0}^{N-1} c_p(n) c_m^*(n) = \sum_{n=0}^{N-1} \exp(j2\pi f_p n/4) \exp(-j2\pi f_m n/4) = \begin{cases} 0 & , p \neq m \text{ , orthogonal} \\ 4 & , p = m \text{ , otherwise} \end{cases}$$

$$1 \leq p \leq 4 \text{ , } 1 \leq m \leq 4 \text{ , } 0 \leq n \leq N-1=3 \quad (4.4)$$

For instance after setting, $p = 1, m = 2$, (4.4) evaluates to

$$\begin{aligned} \sum_{n=0}^{N-1} \exp(j2\pi f_1 n/4) \exp(-j2\pi f_2 n/4) &= \sum_{n=0}^3 \exp(j2\pi n/4) \exp(-j4\pi n/4) \\ &= \sum_{n=0}^3 \exp(-j2\pi n/4) = \overbrace{\exp(0)}^{n=0} + \overbrace{\exp(-j\pi/2)}^{n=1} + \overbrace{\exp(-j\pi)}^{n=2} + \overbrace{\exp(-j3\pi/2)}^{n=3} \\ &= 1 - j - 1 + j = 0 \end{aligned} \quad (4.5)$$

Hence we conclude that with the choice of numeric settings of (4.1) and (4.2), $c_1(n)$ and $c_2(n)$ are indeed orthogonal.

Exercise 4.1 : Prove that the validity of the first line of (4.4) by inserting for the subcarriers from (4.1) and (4.2) for the case other than the one evaluated in (4.5).

Now we consider the operations on the side of the receiver. We recollect from (3.13) and (3.14), in OFDM modulation corresponds to inverse discrete Fourier transform operation, while OFDM demodulation corresponds to discrete Fourier transform operation. Then after modulation, for the simple 4 PSK case, we will have

$$\begin{aligned} y(n) &= \frac{1}{N} \sum_{k=1}^K s_k \exp(j2\pi f_k n/N) = \frac{1}{N} \sum_{k=1}^4 s_k \exp(j2\pi f_k n/N) \\ &= \frac{1}{N} [s_1 \exp(j2\pi f_1 n/N) + s_2 \exp(j2\pi f_2 n/N) + s_3 \exp(j2\pi f_3 n/N) + s_4 \exp(j2\pi f_4 n/N)] \\ &= \frac{1}{N} [s_2 \exp(j2\pi f_1 n/N) + s_2 \exp(j2\pi f_2 n/N) + s_3 \exp(j2\pi f_3 n/N) + s_1 \exp(j2\pi f_4 n/N)] \\ &= \frac{1}{N} [j \exp(j2\pi f_1 n/N) + j \exp(j2\pi f_2 n/N) - \exp(j2\pi f_3 n/N) + \exp(j2\pi f_4 n/N)] \end{aligned} \quad (4.6)$$

On the receiver side, performing the correlation on the first arm of Fig. 3.7, we will get

$$\begin{aligned} d_1 &= \sum_{n=0}^{N-1} y(n) \exp(-j2\pi f_1 n/N) = \\ &= \frac{1}{N} \left[\sum_{n=0}^{N-1} s_1 \exp(j2\pi f_1 n/N) \exp(-j2\pi f_1 n/N) + s_2 \exp(j2\pi f_2 n/N) \exp(-j2\pi f_1 n/N) \right. \\ &\quad \left. + s_3 \exp(j2\pi f_3 n/N) \exp(-j2\pi f_1 n/N) + s_4 \exp(j2\pi f_4 n/N) \exp(-j2\pi f_1 n/N) \right] \\ &= s_1 + 0 + 0 + 0 = s_2 = j \end{aligned} \quad (4.7)$$

As seen from the result of (4.7), demodulation is successful, hence the PSK the symbol placed on the first subcarrier $c_1(n)$ at the transmitter, is correctly recovered as s_2 or j at the receiver.

Exercise 4.2 : Prove that the symbols placed onto other subcarriers, i.e. $c_2(n)$, $c_3(n)$, $c_4(n)$ are also demodulated successfully, as it is done for the first symbol in (4.7).

Exercise 4.3 : Available on the course webpage, the Matlab file OFDM_exp_simple.m implements the modulation and demodulation for an OFDM system whose symbols are obtained from an 8 rectangular QAM constellation. To increase the number of subcarriers to 8, four higher frequencies are selected in addition to those listed in (3.6). Run and observe the outputs of OFDM_exp_simple.m. Find the additional subcarriers, observe the orthogonality of all subcarriers both from the time waveforms and the C_{tmat} matrix written onto the workspace. Comment on the spectrums of modulated subcarriers. Make several runs to see if demodulation is successful. Repeat this by changing the targeted symbol of demodulation on line 44 of the code. By modifying the symbol duration T on line 4 of the code, observe that the orthogonality is destroyed and the transmitted symbol is not demodulated correctly.

References

1. John G. Proakis, Masoud Salehi, "Communication Systems Engineering" 2nd Ed., Prentice Hall 2002, ISBN : 0-13-061793-8, Chapter 9.
2. S. K. Mitra, "Digital Signal Processing", 3rd Ed. McGraw Hill 2006, ISBN : 0-13-084788-7, pp 233 - 244.
3. Matlab help - DFT
4. My own lecture notes.